Kuwait University — Math 101 — July 20th, 2000
Math. & Comp. Sci. Dept. — Second Exam — Time: 75 minutes

Calculators and Mobile Phones are not allowed.

1. Let $y = \tan\theta$.

   a) Use differentials to find dy if θ changes from $45°$ to $44°$.

   b) Use a linear approximation to estimate $\tan 44°$.

   (3 Points)

2. As a right circular cylindrical metal rod is being heated, its height is increasing at a rate of 0.002 cm/min and its radius is increasing at a rate of 0.001 cm/min. At what rate is the volume changing when the rod has height 20 cm and radius 4cm.

   (3 Points)

3. a) State the mean value theorem.

   b) Use the mean value theorem to show that if u and v are any real numbers, then
   $$|\cos^7 u - \cos^7 v| \leq 7|u - v|.$$

   (3 Points)

4. Find two real numbers whose difference is 10 and their product is minimum.

   (3 Points)

5. Evaluate the following integrals:

   a) $\displaystyle\int \frac{2t-5}{\sqrt[3]{t}}dt$, b) $\displaystyle\int \frac{1}{\sin 2x \tan 2x}dx$.

   (3 Points)

6. Let $f(x) = \dfrac{x^2}{4 - x^2}$

   a) Find the vertical and horizontal asymptotes (if any).

   b) Show that $f'(x) = \dfrac{8x}{(4-x^2)^2}$.

   c) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema (if any).

   d) Given that $f''(x) = \dfrac{32 + 24x^2}{(4-x^2)^3}$. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the point of inflection (if any).

   e) Sketch the graph of f.

   (10 Points)

a) $y = f(\theta) = \tan\theta$

$\Delta y \simeq dy = \sec^2\theta \, d\theta = \sec^2\frac{\pi}{4}\cdot\left(-\frac{\pi}{180}\right) = 2\left(-\frac{\pi}{180}\right)$

$\therefore \quad \Delta y \simeq -\frac{\pi}{90}$

b) $f(\theta+\Delta\theta) \simeq f(\theta) + f'(\theta)\Delta\theta$

$\tan 44° = \tan(45°-1°) = \tan\left(\frac{\pi}{4}-\frac{\pi}{180}\right) \simeq \tan\frac{\pi}{4} + \sec^2\frac{\pi}{4}\cdot\left(-\frac{\pi}{180}\right)$

$\therefore \quad \tan 44° \simeq 1 - \frac{\pi}{90}$

2) $r = 4\text{ cm}$, $h = 20\text{ cm}$, $\frac{dr}{dt} = 0.001\text{ cm/min}$, $\frac{dh}{dt} = 0.002\text{ cm/min}$

$V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$

$= 2\pi(4)(20)(0.001) + \pi(16)(0.002)$

$= 0.16\pi + 0.032\pi$

$= 0.192\pi \text{ cm}^3/\text{min}$

3) a) Theorem: (MVT).

b) $f(\theta) = \cos^7\theta$ is continuous on $[u,v]$ and differentiable on (u,v)

by the M.V.T there exist $\xi \in (u,v)$ such that

$$f'(\xi) = \frac{f(u)-f(v)}{u-v} = \frac{\cos^7 u - \cos^7 v}{u-v}$$

$$f'(\theta) = -7\cos^6\theta \cdot \sin\theta$$

$$\left|\frac{\cos^7 u - \cos^7 v}{u-v}\right| = \left|-7\cos^6\xi \sin\xi\right| = \left|-7\cos^6_\xi \sin\xi\right|$$

$$= |-7|\,|\cos^6\xi|\,|\sin\xi| \le 7$$

$$\Rightarrow \left|\cos^7 u - \cos^7 v\right| \le 7\,|u-v|.$$

4) $x - y = 10$

$P = x\cdot y = x(x-10) = x^2 - 10x$

$\frac{dP}{dx} = 2x - 10 = 0 \Rightarrow x = 5$; $\left.\frac{d^2P}{dx^2} = 2\right|_{x=5} > 0 \Rightarrow P$ is minim

$\boxed{x=5 \;\&\; y=-5}$ $\qquad \Rightarrow y = 5-10 = -5$

5) a) $\int \frac{2t-5}{\sqrt[3]{t}}\,dt = \int (2t-5)\,t^{-1/3}\,dt = \int 2t^{2/3}\,dt - \int 5t^{-1/3}\,dt$

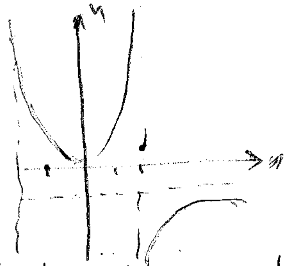
$= 2\cdot\frac{1}{1+\frac{2}{3}}\,t^{\frac{2}{3}+1} - 5\cdot\frac{1}{-\frac{1}{3}+1}\,t^{-1/3+1} + C$

$= \frac{6}{5}t^{5/3} - \frac{15}{2}t^{2/3} + C$

b) $I = \int \frac{1}{\sin 2x \,\tan 2x}\,dx$

$= \int \csc 2x \,\cot 2x \,dx$

$= -\frac{1}{2}\csc 2x + C$



6) $f(x) = \frac{x^2}{4-x^2}$

$D_f = \mathbb{R}\setminus\{-2,2\}$; f is a rational function, it is continuous everywhere in its D_f.

V.A : $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+}\frac{x^2}{4-x^2} = \frac{4}{0^+} = +\infty$

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-}\frac{x^2}{4-x^2} = \frac{4}{0^-} = -\infty$ $\Big\}$ $x=-2$ is a V.A

$\lim\limits_{x\to 2^+} f(x) = \frac{4}{0^-} = -\infty$, $\lim\limits_{x\to 2^-} f(x) = \frac{4}{0^+} = +\infty \Rightarrow x = 2$ is

H.A $\lim\limits_{x\to +\infty} f(x) = -1$ & $\lim\limits_{x\to -\infty} f(x) = -1 \Rightarrow y = -1$ is a H.A

$f'(x) = \frac{2x(4-x^2) + 2x\,x^2}{(4-x^2)^2} = \frac{8x - \cancel{2x^3} + \cancel{2x^3}}{(4-x^2)^2} = \frac{8x}{(4-x^2)^2} = 0 \Rightarrow x =$

x	$-\infty$		-2		0		2		$+\infty$
$8x$		$-$		$-$	0	$+$		$+$	
$(4-x^2)^2$		$+$	0	$+$		$+$	0	$+$	
$f'(x)$		$-$	‖	$-$	0	$+$	‖	$+$	
$f(x)$	-1	↘ $-\infty$	‖	$+\infty$ ↘	0	↗ $+\infty$	‖	$-\infty$ ↗	-1

f is increasing on $(0,2)\cup(2,+\infty)$

f is decreasing on $(-\infty,-2)\cup(-2,$

$(0,0)$ is a local minimum